PUTNAM PRACTICE SET 9

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Problem 1. Let S be the set of real numbers which is closed under multiplication, i.e., if $a, b \in S$ then $ab \in S$. Let T and U be disjoint subsets of S whose union is S. Given that the product of 3 elements of T (not necessarily distinct) is also contained in T, and similarly, the product of 3 elements of U is also contained in U, the prove that at least one of the two sets T or U is closed under multiplication.

Problem 2. Let $x_1(t), \dots, x_n(t)$ be differentiable functions satisfying the following system of differential equations:

$$x_i'(t) = \sum_{j=1}^{n} a_{i,j} x_j(t),$$

for given positive real numbers $a_{i,j}$. If

$$\lim_{t \to \infty} x_i(t) = 0 \text{ for each } i = 1, \dots, n,$$

then prove that the functions $x_1(t), \dots, x_n(t)$ are linearly dependent, i.e., there exist constants $c_1, \dots, c_n$ (not all equal to 0) such that

$$\sum_{i=1}^{n} c_i x_i(t) = 0.$$

Problem 3. Let p be a prime number greater than 3 and let $k = \left[\frac{2p}{3}\right]$, where $[z]$ denotes (as always) the integer part of the real number z (i.e., the largest integer less than or equal to z). Prove that p^2 divides $\sum_{i=1}^{k} \binom{p}{i}$.

Problem 4. Let c be a positive real number. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the property that for each real number x, we have that $f(x) = f(x^2 + c)$.