PUTNAM PRACTICE SET 6

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Problem 1. Let a and s be real numbers satisfying the following properties:

- $0 < a \leq 1$; and
- $s > 0$, but $s \neq 1$.

Prove that $\frac{1-s^a}{1-s} \leq (1+s)^{a-1}$.

Problem 2. Let S be the set of all real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$ where m and n are positive integers. Prove that for each two distinct elements $u < v$ contained in S, there exists another element $w \in S$ such that $u < w < v$.

Problem 3. We consider a set S of finitely many disks in the cartesian plane (of arbitrary centers and arbitrary radii) and we let A be the area of the region represented by their union. Prove that there exists a subset $S_0 \subseteq S$ satisfying the following two properties:

- any two disks from S_0 are disjoint.
- the sum of the areas of the disks from S_0 is at least $\frac{A}{9}$.

Problem 4. Let $\{u_n\}_{n\geq 1}$ be a recurrence sequence defined by $u_{n+1} = \frac{\sqrt[3]{64u_n+15}}{4}$ for each $n \geq 1$. Find $\lim_{n\to\infty} u_n$.