PUTNAM PRACTICE SET 8

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Problem 1. Prove that if $a, b, c \in \mathbb{C}$ and the following relations are satisfied:

- $a + b + c = 0$; and
- $|a| = |b| = |c|$,

then $a^3 = b^3 = c^3$.

Can this result be extended to more than 3 complex numbers?

Solution. Clearly, if $a = 0$, then $b = c = 0$. So, from now on, we assume neither number is 0 and then dividing by a, we may assume we deal with the complex numbers $1, r, s$ with $r = e^{i\alpha}$ and $s = e^{i\beta}$ for some real numbers α, β such that $1 + r + s = 0$, which means

$$\sin(\alpha) + \sin(\beta) = 0 \text{ and so, } \sin(\beta) = -\sin(\alpha),$$

which in particular, yields $\cos(\beta) = \pm\cos(\alpha)$. Also, we need to have $1 + \cos(\alpha) + \cos(\beta) = 0$ and so, this means we cannot have $\cos(\beta) = -\cos(\alpha)$ and instead, $\cos(\beta) = \cos(\alpha) = \frac{-1}{2}$. In conclusion, r and s are the two primitive third roots of unity, which yields that $a^3 = b^3 = c^3$ as desired.

Now, if we deal with more than 3 complex numbers, it will not be enough to assume that $a_1 + \cdots + a_n = 0$ and $|a_1| = |a_2| = \cdots = |a_n|$ in order to conclude that $a_1^n = a_2^n = \cdots = a_n^n$. Indeed, we can let

$$a_k = e^{2\pi ik/(n-2)} \text{ for } k = 1, \dots, n-2$$

and $a_n = -a_{n-1}$ for some complex number a_{n-1} which is not a root of unity.

Problem 2. If the series $\sum_{n=1}^{\infty} a_n$ of real numbers converges, does $\sum_{n=1}^{\infty} a_n^3$ converge?

Solution. No; here's a counterexample. For each positive integer n, we let $a_{2^n} = \frac{1}{\sqrt[3]{n}}$. Now, for each $n \geq 1$ and for each $1 \leq k < 2^n$, we let $a_{2^n+k} = -\frac{1}{\sqrt[3]{n}\cdot 2^n}$. For completion, we let $a_1 = 0$.

Claim 1. The series $\sum_{k=1}^{\infty} a_k$ converges.

Proof of Claim 1. For each $1 < \ell < m$, we let n_1 be the unique positive integer such that $2^{n_1} \leq \ell < 2^{n_1+1}$ and also, we let n_2 be the unique positive integer such that $2^{n_2} \leq m < 2^{n_2+1}$; then we let $k_1 := \ell - 2^{n_1}$ and $k_2 := m - 2^{n_2}$. Clearly, $1 \leq n_1 \leq n_2$. We have

$$\left|\sum_{k=\ell}^{m} a_k\right| < \frac{1}{\sqrt[3]{n_1}} + \frac{1}{\sqrt[3]{n_2}} + \sum_{i=n_1+1}^{n_2-1} \frac{1}{2^i \cdot \sqrt[3]{i}}$$

and so, if $n_1, n_2 > N$, then

$$\left|\sum_{k=\ell}^{m} a_k\right| < \frac{2}{\sqrt[3]{N}} + \frac{1}{2^N} \to 0 \text{ as } N \to \infty.$$

So, indeed, $\sum_{k=1}^{\infty} a_k$ converges.

Claim 2. The series $\sum_{k=1}^{\infty} a_k^3$ diverges.

Indeed, for each $n \geq 1$ and for each $0 \leq k \leq 2^n - 1$, we have

$$\sum_{k=1}^{2^n} a_k^3$$

$$> \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{2^i}{i \cdot 8^i}$$

$$> \sum_{i=1}^{n} \frac{1}{2i},$$

which diverges to ∞, thus proving that $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 3. For what pairs (a, b) of positive real numbers we have that the integral

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) \mathrm{dx}$$

converges.

Solution. The key observation is that

$$\sqrt{x+a} - \sqrt{x} = \frac{a}{\sqrt{x+a} + \sqrt{x}} = \frac{a}{2\sqrt{x}} - \frac{a^2}{2\sqrt{x} \cdot \left(\sqrt{x} + \sqrt{x+a}\right)^2};$$

in other words,

$$\left| \sqrt{x+a} - \sqrt{x} - \frac{a}{2\sqrt{x}} \right| < \frac{a^2}{2x^{\frac{3}{2}}}.$$

So, $\sqrt{\sqrt{x+a} - \sqrt{x}} = \sqrt{\frac{a}{2}} \cdot \frac{1}{\sqrt[4]{x}} + f_a(x)$, where $|f_a(x)| < C_a x^{-5/4}$ for some positive constant C_a depending only on a (and independent of x). A similar computation yields that

$$\sqrt{\sqrt{x} - \sqrt{x-b}} = \sqrt{\frac{b}{2}} \cdot \frac{1}{\sqrt[4]{x}} + f_b(x),$$

where $|f_b(x)| < C_b x^{-5/4}$ for some positive constant C_b depending only on b (and independent of x). This means that

$$\int_b^{\infty} |f_a(x) - f_b(x)| \, \mathrm{dx} < \int_b^{\infty} |f_a(x)| \mathrm{dx} + \int_b^{\infty} |f_b(x)| \mathrm{dx} < \infty.$$

So, we conclude that $\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) \mathrm{dx}$ converges if and only if $\int_b^{\infty} \frac{\sqrt{a} - \sqrt{b}}{\sqrt{2} \cdot \sqrt[4]{x}} \mathrm{dx}$ converges, which happens if and only if $a = b$.

Problem 4. For each $n \in \mathbb{N}$, we let S_n be the set of all pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with the property that $x^3 - 3xy^2 + y^3 = n$.

(a) For each $n \in \mathbb{N}$, prove that either S_n is the empty set, or it has at least 3 elements.
(b) Prove that S_{2021} is the empty set.

Solution.

(a) We observe that once (x, y) is a solution, then also $(-y, x-y)$ is a solution and therefore, also $(y-x, -x)$ is a solution; finally, applying the transformation $(x, y) \mapsto (-y, x-y)$ to the last solution $(y-x, -x)$, we recover the original solution (x, y). We note that for a solution (x, y), the other two solutions $(y-x, -x)$ and $(-y, x-y)$ are distinct because otherwise we would have $y = -x$ and $x = y - x = -x - x$, i.e., $x = 0$ and so, $y = 0$, contradicting the fact that $x^3 - 3xy^2 + y^3 = n$ cannot have the trivial solution. So, indeed, once there exists a solution, then there are at least 3 solutions.

The idea for this solution comes from looking at transformations of the form $(x, y) \mapsto (ax+by, cx+dy)$ which preserve the quantity $x^3 - 3xy^2 + y^3$; also, we search for small values for a, b, c, d.

(b) Using Fermat's Little Theorem, we have $x^3 \equiv x \pmod 3$ and so,

$$2 \equiv 2021 \equiv x^3 - 3xy^2 + y^3 \equiv x + y \pmod 3$$

and so, noting part (a) above, we may assume x is divisible by 3 and therefore, $y \equiv 2 \pmod 3$. But then $y^3 \equiv 8 \pmod 9$ and overall (because 3 divides x),

$$x^3 - 3xy^2 + y^3 \equiv 8 \not\equiv 2021 \pmod 9,$$

contradiction. Therefore, there are no solutions to $x^3 - 3xy^2 + y^3 = 2021$.