The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations

January 2022

---

1. (8 points) (a) (2 points) Find a basis for the space M_2 of 2×2 matrices.
   (b) (2 points) What is the dimension of M_2?
   (c) (2 points) Find a basis for the subspace D_2 of diagonal 2×2 matrices.
   (d) (2 points) What is the dimension of D_2?

2. (12 points) Let $x_1 = (-1, 1)$ and $x_2 = (1, 0)$ and consider the sets $S_1, S_2 \subset \mathbb{R}^2$ given by

$$S_1 = \{v : v^T x_1 = 0\}$$

and

$$S_2 = \{v : v^T x_2 = 0\}.$$

   (a) (4 points) Find an expression for the projection operators P_1 onto S_1 and P_2 onto S_2.
   (b) (4 points) Let $y = (1, 2)$. Compute $P_2 y$, the projection of y onto S_2, and $P_1 P_2 y$, and $P_2 P_1 P_2 y$.
   (c) (4 points) Find $\lim_{n\to\infty} (P_1 P_2)^n y$. Justify your answer.

3. (10 points) Let M be a square matrix of size $n \times n$ with distinct eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Show that the determinant of M is the product of its eigenvalues: $\det(M) = \prod_{i=1}^{n} \lambda_i$.

4. (10 points) (a) *[7 points]* Let n be a non-negative integer. **Legendre's differential equation** is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \quad -1 \le x \le 1,$$

and has the solution, $y(x) = P_n(x)$, which is regular at $x = \pm 1$ and satisfies

$$P_n(1) = 1 \qquad \& \qquad \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}.$$

Look for a solution to the differential equation with the form of a power series,

$$y = \sum_{j=0}^{\infty} c_j x^j.$$

Establish the recurrence relation satisfied by the coefficients c_j, and thereby argue that the solutions $P_n(x)$ take form of polynomials of finite degree if $n = 0, 1, 2, ...$ Hence find $P_n(x)$ for $n = 0, 1, 2, 3$.

(b) *[3 points]* Place Legendre's equation in Sturm-Liouville form, and then show that

$$\int_0^1 P_n(x)dx = \frac{P_n'(0)}{n(n+1)}$$

for $n > 0$. Hence use Sturm-Liouville theory to establish that

$$\text{sgn}(x) = \sum_{n\ odd} \frac{(2n+1)P_n'(0)}{n(n+1)} P_n(x)\,,$$

where $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$.

5. (10 points) By either observing that $y(x) = x$ is a homogeneous solution to the differential equation or otherwise, solve the differential equation

$$x^2 y'' - \alpha(xy' - y) = x(1+x), \qquad y(0) = y(1) = 0,$$

where α is a positive parameter. Is the solution unique?

6. (10 points) Use separation of variables to solve Laplace's equation

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

outside the unit disk, $r \ge 1$, subject to

$$u(1,\theta) = \begin{cases} 0 & \theta = 0 \text{ or } \pi \\ \pi/2 & 0 < \theta < \pi \\ -\pi/2 & \pi < \theta < 2\pi \end{cases}.$$

Using

$$\frac{1}{2}\ln\left(\frac{1+\psi}{1-\psi}\right) = \sum_{n>0, n\ odd} \frac{\psi^n}{n},$$

sum the series for $u(r,\theta)$, and hence write down a compact logarithmic expression for the solution.