The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 2022

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1. (8 points) Find the shortest distance from x to $U = \text{span}\{u_1, u_2\} \subseteq \mathbb{R}^4$ where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

2. (8 points) Let A be a real 3×3 matrix and suppose that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors of A. Show that A is symmetric.

3. (14 points) Recall the matrix norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.

   (a) (7 points) Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and singular values $\sigma_1, \ldots, \sigma_n$. What is $\|A\|$? Justify your answer.

   (b) (7 points) Determine the matrix norm $\|A\|$ for the matrix

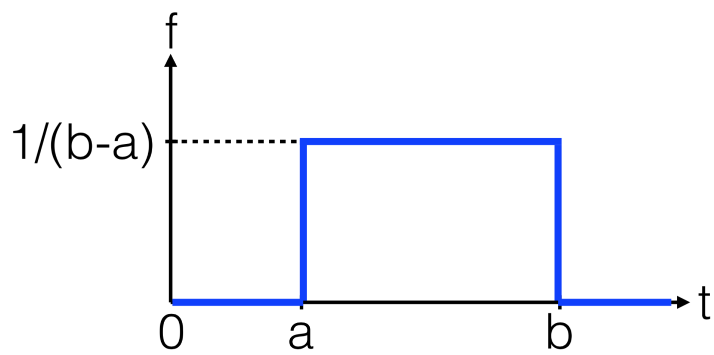
$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

4. (10 points) We consider the following ODE initial-value problem:

$$x'' + \gamma x = f(t), \quad x(0) = 0, \quad x'(0) = 1, \tag{1}$$

where γ is a non-zero positive parameter and $f(t)$ is an external forcing.

(a) *[5 points]* We assume that $f(t)$ is a switch on-switch off function (also known as rectangular function or top hat function) between $t = a$ and $t = b > a$ and of intensity $\frac{1}{b-a}$ shown in the picture below.



Solve the problem using Laplace transform and write the solution as a function of γ, a and b. Denote this solution $x_S(t)$.

(b) *[2 points]* We assume that $f(t) = \delta(t-a)$ is a delta Dirac function at $t = a$. Solve the problem using Laplace transform and write the solution as a function of γ and a. Denote this solution $x_D(t)$.

(c) *[3 points]* Take $\gamma = 1$ for simplicity. Using $x_S(t)$ and $x_D(t)$ from question (a) and question (b) respectively, establish that

$$\lim_{b \to a} \frac{\cos(t-a)u(t-a) - \cos(t-b)u(t-b)}{b-a} = \delta(t-a) - \sin(t-a)u(t-a)$$

where $u(t-x)$ is the Heaviside function defined as

$$u = \begin{cases} 0\,, & t < x \\ 1\,, & t \geq x \end{cases}$$

Help with Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1. $\sin(xt)$	$\dfrac{x}{s^2+x^2}, \quad s > 0$
2. $\cos(xt)$	$\dfrac{s}{s^2+x^2}, \quad s > 0$
3. $u(t-x)f(t-x)$	$e^{-xs}F(s)$
4. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$

5. (10 points) (a) *[7 points]* Apply the method of separation of variables to determine the solution to the one dimensional time-dependent heat equation subject to the following periodic boundary conditions and initial condition:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } \ t > 0, \ \ 0 \le x \le 2,$$

$$\text{BC:} \quad u(0,t) = u(2,t), \ \text{ and } \ \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(2,t)}{\partial x}$$

$$\text{IC:} \quad u(x,0) = f(x) = \begin{cases} 1-x & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 2 \end{cases}$$

(b) *[3 points]* Use the Fourier series you found in (a) and by evaluating the value of the periodic extension of $f(x)$ at $x=0$, find the sum of the infinite series

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

6. (10 points) We consider the following Poisson problem in the bounded 2-D domain Ω with Neumann boundary conditions on the boundary of Ω denoted by $\partial\Omega$:

$$\Delta u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega$$

where n is the unit normal vector over $\partial\Omega$, f and g are two functions of the spatial coordinates.

(a) *[2 points]* Discuss the uniqueness of the solution u.

(b) *[3 points]* Find a condition on f and g for the problem to be well-posed (and therefore for the solution u to exist).

(c) *[5 points]* We take $\Omega = [0,\pi] \times [0,\pi]$, $f = \delta(x_1 - \overline{x_1}, x_2 - \overline{x_2})$ and $g = 0$, where δ is the delta Dirac function centered at $(x_1, x_2) = (\overline{x_1}, \overline{x_2}) \in [0,\pi] \times [0,\pi]$. Find the solution $u(x_1, x_2)$ by using an eigenfunction expansion and separation of variables.